MOTION OF A VISCOUS THERMALLY CONDUCTING GAS UNDER THE EFFECT OF A ROTATING DISK

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UDC 532.516

INTRODUCTION

The problem of the steady laminar motion of an incompressible viscous fluid under the effect of an unbounded disk rotating with constant angular velocity was first formulated and solved approximately by Karman [1]. The solution of this problem, constructed for the Navier–Stokes equations, at the same time displays a number of features characteristic of flows in boundary layers. It is not hard to establish, for example, that Karman's solution corresponds to a finite displacement thickness which does not depend on the radial coordinate, and one can introduce other thicknesses defined in boundary-layer theory. In this connection many of the methods of boundary-layer theory are applied in the solution of the Karman problem and its modifications [2-6].

Eventually, attempts were made to construct a solution analogous to Karman's solution for the more general case of a compressible, viscous, and thermally conducting fluid [7, 8]. In [7], in particular, the flow of a gas under the effect of a rotating disk with a constant surface temperature was analyzed in the boundary-layer approximation. However, the construction of higher approximations to the solution of the total problem, determined by the method of joinable asymptotic expansions [9], is affected by the presence of secular terms which are proportional to ever-higher powers of the radial coordinate, which indicates divergence of the approximations. Thus, it becomes clear that in the case of a constant surface temperature the problem of an unbounded disk rotating in a gas does not have a solution in the boundary-layer approximation and requires the complete Navier–Stokes equations for its description for any values of the characteristic parameters.

In order to have the possibility of applying the perturbation method to the solution of the problem of a disk in a compressible fluid one must modify the formulation of the problem, assuming that the surface temperature varies with the radius. On the basis of the assumption that the radial temperature variation obeys a power law, a transformation of the original system of Navier–Stokes equations to dimensionless form, which is most suitable for the transition to boundary-layer equations, is performed below. It is shown that such a transition is possible only for a quadratic law of radial temperature variation, when there exists a self-similar solution of the complete Navier–Stokes equations. By introducing into the analysis the small parameter ε, which is analogous to $\mathrm{Re}^{-1/2}$ in problems of streamline flow, and passing to the limit $\varepsilon \to 0$, we obtain the correct variant of the equations of the boundary layer in a gas at a rotating disk. A study of these equations indicates the presence of some special features of such a boundary layer which are not encountered in ordinary boundary layers. For example, it is ascertained that a solution of the boundary-layer equations can exist if outside the boundary layer there is adiabatic motion of a certain class with radially varying pressure and temperature and with a constant angular velocity of rotation of the entire mass of gas. The properties of the attenuation of perturbations with greater distance from the surface are also analyzed and the results of a numerical calculation of the flow parameters in the boundary layer are presented for a certain concrete example.

1. Suppose a disk of infinitely large radius, rotating with a constant angular velocity ω, is located in the plane $z = 0$. The half-space above the disk is filled with a viscous thermally conducting gas which has constant specific heat capacities c_p and c_v, a constant Prandtl number σ, an equation of state in the Clapeyron form, and a power-law dependence of the viscosity on the temperature. Under the effect of the

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 92-100, May-June, 1975. Original article submitted July 24, 1974.

rotation of the disk the gas is set into motion and, taking this motion as established and axisymmetric, we use for its description the Navier–Stokes equations, which in the cylindrical coordinate system have the form

$$\frac{\partial(r\rho v_r)}{\partial r}+\frac{\partial(r\rho v_z)}{\partial z}=0;\quad p=(\varkappa-1)\rho e;\quad \mu=Ae^n; \tag{1.1}$$

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta^2}{r}\right)=-\frac{\partial p}{\partial r}+\frac{2}{3}\frac{\partial}{\partial r}\left\{\mu\left[3\frac{\partial v_r}{\partial r}-\right.\right.$$

$$\left.\left.-\frac{1}{r}\frac{\partial(rv_r)}{\partial r}-\frac{\partial v_z}{\partial z}\right]\right\}+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right]+2\mu\frac{1}{r}\left(\frac{\partial v_r}{\partial r}-\frac{v_r}{r}\right),$$

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_rv_\theta}{r}\right)=\frac{\partial}{\partial z}\left(\mu\frac{\partial v_\theta}{\partial z}\right)+\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)\right]+2\mu\frac{1}{r}\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right);$$

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+v_z\frac{\partial v_z}{\partial z}\right)=-\frac{\partial p}{\partial z}+\frac{2}{3}\frac{\partial}{\partial z}\left\{\mu\left[2\frac{\partial v_z}{\partial z}-\frac{1}{r}\frac{\partial(rv_r)}{\partial r}\right]\right\}+\frac{1}{r}\frac{\partial}{\partial r}\left[r\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\right];$$

$$\rho\left(v_r\frac{\partial e}{\partial r}+v_z\frac{\partial e}{\partial z}\right)+p\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)=\frac{\varkappa}{\sigma}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu\frac{\partial e}{\partial r}\right)+\right.$$

$$\left.+\frac{\partial}{\partial z}\left(\mu\frac{\partial e}{\partial z}\right)\right]+\mu\left\{2\left[\left(\frac{\partial v_r}{\partial r}\right)^2+\frac{v_r^2}{r^2}+\left(\frac{\partial v_z}{\partial z}\right)^2\right]+\left(\frac{\partial v_\theta}{\partial z}\right)^2+\right.$$

$$\left.+\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)^2+\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)^2\right\}-\frac{2}{3}\mu\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)^2,$$

where $\varkappa = c_p/c_V$, $e = c_VT$ is the internal energy, A and n are constant parameters, and the remaining notation coincides with that in standard use.

In order to make it possible to consider a certain class of problems with different boundary conditions, we assume that at the characteristic surface which we have chosen the temperature, pressure, and density obey the following laws:

$$e_*\equiv c_vT_*=B_Tr^k;\quad p_*=(\varkappa-1)B_TB_\rho r^{kn};\quad \rho_*=B_\rho r^{k(n-1)}, \tag{1.2}$$

where B_T, B_ρ, and k are assigned constants. The choice of the exponent in the law of variation of p_* is dependent on the convective term of the energy equation from system (1.1) having the same order with respect to r as the term containing the thermal conduction.

From (1.2) and (1.1) one can obtain an expression for the characteristic kinematic viscosity:

$$\nu_*=AB_T^nB_\rho^{-1}r^k,$$

after which the characteristic Reynolds number is represented in the form

$$\mathrm{Re}=\frac{\omega r^2}{\nu_*}=\varepsilon^{-2}r^{2-k},\quad \varepsilon=\left(\frac{AB_T^n}{\omega B_\rho}\right)^{1/2}, \tag{1.3}$$

where the value ε, which is dimensional in general, is introduced for convenience of notation.

Let us transform the system of equations (1.1) to dimensionless form, using Eqs. (1.2) and (1.3) for this. As seen from (1.3), the radial coordinate r plays the role of the characteristic length here, while the scales of the lengths and velocities in the axial direction must be chosen in the same way as is done in boundary-layer theory, i.e.

$$z_*=\frac{r}{\sqrt{\mathrm{Re}}}=\sqrt{\frac{\nu_*}{\omega}}=\varepsilon r^{k/2};$$

$$v_{z*}=\frac{\omega r}{\sqrt{\mathrm{Re}}}=\varepsilon\omega r^{k/2}.$$

After the transformation the equations have the form

$$r = \varepsilon^{\frac{2}{k-2}} \xi^{\frac{2}{k-2}};\quad z = \varepsilon r^{k/2}\vartheta;\quad v_r = \omega r F;$$
$$v_\theta = \omega r G;\quad v_z = \varepsilon\omega r^{k/2} N;\quad e = B_T r^k Q;$$
$$p = (\varkappa - 1) B_T B_\rho r^{kn} P;\quad \rho = B_\rho r^{k(n-1)} D;$$
$$\mu = B_\rho \omega \varepsilon^2 r^{kn} Q^n, \tag{1.4}$$

where F, G, N, Q, P, and D are functions of the two new arguments ξ and ϑ. To shorten the notation we introduce the differential operator symbol

$$\Delta^* = -\frac{k}{2}\vartheta\frac{\partial}{\partial\vartheta} + \frac{k-2}{2}\xi\frac{\partial}{\partial\xi} \tag{1.5}$$

and define $\gamma = B_T\omega^{-2}$. The substitution of Eq. (1.4) into Eqs. (1.1) gives

$$P = QD,\quad (kn - k + 2) DF + \Delta^*(DF) + \partial(DN)/\partial\vartheta = 0; \tag{1.6}$$
$$D(F^2 - G^2 + F\cdot\Delta^* F + N\partial F/\partial\vartheta) = -(\varkappa-1)\gamma\varepsilon^{-2}\xi^2(knP +$$
$$+ \Delta^* P) + \frac{2}{3}\xi^2(kn + \Delta^*)[Q^n(F + 2\cdot\Delta^* F - \partial N/\partial\vartheta)] +$$
$$+ \frac{\partial}{\partial\vartheta}\left[Q^n\left(\frac{\partial F}{\partial\vartheta} + \xi^2\cdot\Delta^* N\right)\right] + 2\xi^2 Q^n\cdot\Delta^* F;$$
$$D\left(2FG + F\cdot\Delta^* G + N\frac{\partial G}{\partial\vartheta}\right) = \frac{\partial}{\partial\vartheta}\left(Q^n\cdot\frac{\partial G}{\partial\vartheta}\right) + \xi^2(kn + \Delta^*)(Q^n\cdot\Delta^* G) + 2\xi^2 Q^n\cdot\Delta^* G;$$
$$D(F\cdot\Delta^* N + N\partial N/\partial\vartheta) = -(\varkappa-1)\gamma\varepsilon^{-2}\partial P/\partial\vartheta +$$
$$+ \frac{2}{3}\frac{\partial}{\partial\vartheta}\left[Q^n\left(2\frac{\partial N}{\partial\vartheta} - 2F - \Delta^* F\right)\right] + \left(kn - \frac{k}{2} + 2 + \Delta^*\right)\left[Q^n\left(\frac{\partial F}{\partial\vartheta} + \xi^2\cdot\Delta^* N\right)\right];$$
$$D[F(k + \Delta^*)Q + N\partial Q/\partial\vartheta] + (\varkappa-1)P(2F + \Delta^* F + \partial N/\partial\vartheta) =$$
$$= \frac{\varkappa}{\sigma}\xi^2\{(kn + k + \Delta^*)[Q^n(k + \Delta^*)Q]\} + \frac{\varkappa}{\sigma}\frac{\partial}{\partial\vartheta}\left(Q^n\frac{\partial Q}{\partial\vartheta}\right) +$$
$$+ \gamma^{-1}\varepsilon^2 Q^n\Big\{2[(F + \Delta^* F)^2 + F^2 + (\partial N/\partial\vartheta)^2] + \xi^{-2}(\partial G/\partial\vartheta)^2 +$$
$$+ \xi^{-2}\left(\frac{\partial F}{\partial\vartheta} + \xi^2\cdot\Delta^* N\right)^2 + (\Delta^* G)^2 - \frac{2}{3}\left[(2 + \Delta^*)F + \frac{\partial N}{\partial\vartheta}\right]^2\Big\}.$$

By comparing Eqs. (1.3) and (1.4) one can see that $\xi = \mathrm{Re}^{-1/2}$. If one assumes that the conditions of the given problem allow one to extend to it the principles which are fulfilled in external streamline flow then the limiting transition $\xi \to 0$ should lead to the conversion of Eqs. (1.6) into the equations of the boundary layer at the surface of the rotating disk. However, such a limiting transition does not occur painlessly for all the equations of system (1.6). Turning to the last equation of this system (the energy equation), one can see that formally the main term of this equation as $\xi \to 0$ must be considered as a dissipative term containing the factor ξ^{-2}. But the assumption that the role of dissipation is predominant cannot have a physical basis, while from the point of view of the mathematics one would have to discard all terms containing higher derivatives in such a case, which would lead to the impossibility of solving the boundary problems. The contradiction which thus arises can only indicate one thing: in the general case of the movement of a viscous and thermally conducting gas under the effect of a rotating disk of unbounded dimensions it is impossible to obtain a simplification of the mathematic description of the problem through a transition to boundary-layer equations, and no matter what the mode of motion, one can always find in the stream a region for the study of which one must use the complete Navier–Stokes equations.

The above discussion is valid both for the case of a constant temperature of the disk surface ($k = 0$) and for any power law corresponding to Eqs. (1.2) with $k \neq 2$. The case of $k = 2$ is special and not only leads to a number of simplifications but also allows one to remove the above-mentioned contradiction. In fact, it is seen from (1.3) that with $k = 2$ the Reynolds number is converted to a constant $\mathrm{Re} = \varepsilon^{-2}$, so that $\xi \equiv \varepsilon$ also, and the differential operator (1.5) degenerates in this case into the ordinary operator

$$(\Delta^*)_{k=2} = -\vartheta d/d\vartheta, \tag{1.7}$$

and, consequently, system (1.6) becomes a system of ordinary differential equations. But the most important property of Eqs. (1.6) with k = 2 consists in the fact that the transition $\xi \to 0$ in this case gives $\varepsilon \to 0$ and $\varepsilon/\xi \to 1$, as a result of which the dissipative term of the energy equation proves to have the same order as the term containing the thermal conduction, and no contradictions arise in the transition from the Navier–Stokes equations to the boundary-layer equations.

2. Let us examine in more detail that variant of the motion of a viscous gas above a rotating disk where the surface temperature varies in the radial direction according to a quadratic law, i.e., where Eqs. (1.2) with the particular value k = 2 are applicable. It must be noted that the thermal boundary layer at a disk in an incompressible fluid has been studied in [10] for an arbitrary law of radial temperature variation, while the special case of a quadratic law was analyzed in [11]. In studying the motion of a compressible gas Sychev [12] indicated the possibility of decreasing the number of arguments in the problem of the nonsteady motion under the effect of a rotating conical surface with a linear law of the temperature dependence of the viscosity and with a quadratic law of the radial temperature distribution. Thus, in the case of n = 1 this derivation also extends to the problem being considered here, which confirms the possibility of obtaining a self-similar solution, mentioned in Sec. 1.

Without imposing any restrictions on the value of n we can use the general dimensionless equation (1.6), where one must only set k = 2 and $\xi = \varepsilon$ and take the operator Δ^* as expressed in accordance with (1.7). The limiting transition $\varepsilon \to 0$ allows one to obtain the equations for the boundary layer at a disk rotating in a gas (differentiation with respect to ϑ is denoted by a prime):

$$\begin{gathered}2nDF - \vartheta(DF)' + (DN)' = 0;\quad P = QD;\\ D(NF' - \vartheta FF' + F^2 - G^2) = -(\varkappa - 1)\gamma 2nP + (Q^nF');\\ D(NG' - \vartheta FG' + 2FG) = (Q^nG')',\quad P' \equiv 0;\\ D(NQ' - \vartheta FQ' + 2FQ) + (\varkappa - 1)P(2F - \vartheta F' + N') = \frac{\varkappa}{\sigma}(Q^nQ')' + \frac{1}{\gamma}Q^n(F'^2 + G'^2).\end{gathered} \tag{2.1}$$

Before we formulate the boundary conditions, we should note that one cannot obtain a solution for Eqs. (2.1) with the conditions $P(\infty) = Q(\infty) = 0$ since the condition of constancy of the function P across the layer leads to the trivial and physically meaningless expressions $P \equiv 0$ and $Q \equiv 0$. But if $P = P(\infty) \neq 0$ then at an infinite distance from the disk the function $G(\vartheta)$ should approach some constant limit, i.e., rotational motion of the gas with a certain constant angular velocity must exist far from the surface.

Taking these remarks into account, the boundary conditions for the solution of the problem of the boundary layer at the disk can be represented in the form

$$\begin{gathered}G(0)=1;\ F(0)=N(0)=0;\\ Q(\infty)=1;\ P(\infty)=1;\ F(\infty)=0;\\ G(\infty)=\sqrt{2n(\varkappa-1)\gamma}=U.\end{gathered} \tag{2.2}$$

With such an assignment of the boundary conditions the value $Q(0) = Q_W$, which characterized the temperature distribution over the surface of the disk, will be found in the course of the solution.

The form of Eqs. (2.1) can be simplified in the case of n = 1 if in place of ϑ we introduce the new argument

$$\zeta = \int_0^\vartheta D\,d\vartheta$$

and define

$$W = D(N - \vartheta F).$$

As a result, we obtain

$$\begin{gathered}3F + W' = 0;\ QD = 1;\\ WF' + F^2 - G^2 = -QU^2 + F'';\\ WG' + 2FG = G'';\\ \varkappa WQ' + 2FQ = \frac{\varkappa}{\sigma}Q'' + \frac{2(\varkappa-1)}{U^2}(F'^2 + G'^2).\end{gathered} \tag{2.3}$$

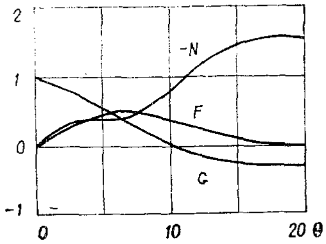

Fig. 1

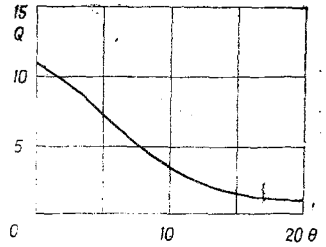

Fig. 2

The boundary conditions (2.2) remain in the previous form, only setting n = 1 and replacing N by W in them.

The system of equations (2.3) is very similar to the equations describing the motion of an incompressible fluid under the effect of a rotating disk [2]. The main distinction of system (2.3) consists in the fact that the energy equation cannot be separated from the equations of motion and the continuity equation, i.e., the thermal boundary layer cannot be analyzed independently from the dynamic boundary layer, which is to be expected in a study of motion and heat transfer in a compressible gas.

Let us study the asymptotic behavior of the solution of Eqs. (2.3) as $\zeta \to \infty$. We will assume that $W(\infty) = -c$, assuming thereby that, as in the case of an incompressible fluid, inflow of fluid toward the disk occurs far from the surface, compensating for the carrying away of mass in the radial direction owing to the effect of the centrifugal forces of inertia. The asymptotic equations for the principal variables can be written in the form

$$F = C_1 e^{-b\zeta}; \quad G = U + C_2 e^{-b\zeta};$$
$$W = -c + \frac{3}{b} C_1 e^{-b\zeta}; \quad Q = 1 + C_3 e^{-b\zeta}. \tag{2.4}$$

Substitution of (2.4) into Eqs. (2.3) leads to algebraic equations for the coefficients C_i which allow one to obtain a nontrivial solution only for a decrement in the attenuation b satisfying the equation

$$(b^2 - bc)^2 = U^2 \left(\frac{2\sigma}{\varkappa} \frac{b - c}{b - \sigma c} - 4 \right). \tag{2.5}$$

The solution of Eq. (2.5) cannot be represented in analytical form for arbitrary $\varkappa$, σ, and U. It is obvious that the case of $\sigma = 1$ and $\varkappa > 0.5$ is special, since in this case Eq. (2.5) does not have real roots. One can show that one of the two existing pairs of complex-conjugate roots necessarily has a positive real part and, consequently, that the system (2.3) under study has a solution which dies out at infinity, but the nature of the dying out proves to be oscillatory in this case.

We can show that in the general case of an arbitrary Prandtl number Eq. (2.5) with any real and nonzero values of U and c has at least one positive root. For this we introduce

$$S = (b - c)/(b - \sigma c)$$

and replace the unknown b with s in Eq. (2.5). We obtain

$$\Phi(s) = 0;$$
$$\Phi = (\sigma s - 1)^2 s^2 - B^2 (s - 1)^4 (s - 2\varkappa\sigma^{-1}); \tag{2.6}$$
$$B^2 = \frac{2\varkappa}{\sigma} \frac{U^2}{(1-\sigma)^2 c^2}.$$

It is necessary to find a real root of Eq. (2.6) satisfying the condition $s > 2\varkappa\sigma^{-1}$, since when this condition is not satisfied the same situation arises as when $\sigma = 1$. There must necessarily be such a root for an equation of the type of (2.6), which with the indicated limitation has two intrinsically positive terms, the second of which has the multiplier -1, is reduced to zero at the starting point of the interval under consideration, but increases in absolute value faster than the first term because it contains a higher power of s. In finding the root s_1 one can also find the corresponding decrement b_1, which is uniquely connected with s_1:

$$b_1 = c(\sigma s_1 - 1)/(s_1 - 1). \tag{2.7}$$

For ordinary gases $\varkappa > 1/2$ and $\sigma < 1$, but it then follows from Eq. (2.7) that $0 < b_1 < c$. Assuming that the function $W(\zeta)$ decreases monotonically as ζ increases, one must take the coefficient C_1 in Eqs. (2.4) as positive. On the other hand, the substitution of Eqs. (2.4) into Eqs. (2.3) allows one to obtain, in particular,

$$C_2 = 2UC_1/(b^2 - bc). \tag{2.8}$$

The sign of the coefficient C_2 itself determines the sign of the derivative $dG/d\zeta$ far from the surface. It is natural to assume that the circular velocity of the gas falls off monotonically with greater distance from the surface, and then $C_2 > 0$. But because of the limitations $C_1 > 0$, $b > 0$, and $b < c$ Eq. (2.8) gives a positive value of C_2 only if $U < 0$. This nonrigorous argument leads to the unexpected result that the rotational motion of the gas far from the surface of the disk can occur in the direction opposite to the rotation of the disk itself.

The qualitative study of the properties of the motion of a viscous gas above a rotating disk which was carried through above pertained only to the limiting mode ($\varepsilon \to 0$) of the boundary layer. If we wish to consider the motion for arbitrary $\varepsilon < 1$ we must turn back to Eqs. (1.6) and use the method of joinable asymptotic expansions [9] mentioned earlier. With $k = 2$ and $\xi \equiv \varepsilon$ and with allowance for (1.7) Eqs. (1.6) serve as the basis for the construction of inner expansions, whereas before the construction of outer expansions one must make the substitutions

$$\vartheta = \vartheta_0/\varepsilon; \quad N = N_0/\varepsilon$$

and operate with the variables ϑ_0 and N_0 in the outer region.

The solution of the boundary problem for Eqs. (2.1) with the conditions (2.2) should be considered as the initial approximation for the inner region. The initial approximation for the outer region will be rotation of the mass of gas as a solid body with a constant angular velocity $\Omega = \omega U$, with the pressure and density corresponding to Eqs. (1.2) with the same value $k = 2$. The construction of all subsequent approximations, both inner and outer, does not raise any fundamental difficulties because of the absence of any obscurities of either a mathematical or a physical nature. The qualitative properties of the flow brought out within the framework of boundary-layer theory should not undergo significant changes when the higher approximations are taken into account. We only need to mention the necessity of modifying the boundary conditions to allow for such effects as slippage, the creep velocity, and the temperature jump, which are proportional to ε or ε^2. A more detailed study of the higher approximations in the solution of the problem of the motion of a gas above a rotating disk goes beyond the scope of the present article.

3. As an example, numerical calculations were made of the boundary layer at a rotating disk for the case of $n = 1$, when one can use the simpler equations (2.5) with the modified boundary conditions (2.2). The other parameters are taken as follows: $\varkappa = 1.4$; $\sigma = 0.7$; $U^2 = 0.09108$. The value of Q_W and the sign of the value $U = v_{\theta\infty}/(\omega r)$ were determined in the course of the solution.

The method of calculation does not differ in principle from that which was described in [13] and comes down to the numerical solution of the Gauchy problem with arbitrariness in the initial values, which joins at some point with the solution which is valid far from the surface and is determined by several terms of the asymptotic expansion; the unknown parameters are made more exact by Newton's method. The fact that the order of system (2.5) is higher than the order of the system solved in [13] does not have essential importance, although it somewhat complicates the algorithm of the solution.

Profiles of the variation in the dimensionless values characterizing the three velocity components and the temperature of the gas in the boundary layer which were obtained as a result of the calculations are shown in Figs. 1 and 2. The graphs confirm the correctness of the conclusions drawn earlier in the qualitative study. In particular, with the value of the modulus U chosen here it proved possible to construct a continuous solution only for $U < 0$, i.e., for the case of rotational motion of the mass of gas far from the disk in the direction opposite to the rotation of the disk itself.

In the case of $n = 1$ under consideration the gas density at the outer limit of the boundary layer is constant: $(B_\rho)_{n=1} = \rho_\infty$. Calculating for the parameters assigned above the mass flow rate of gas flowing in toward the disk in the axial direction, i.e., the radial flow rate of mass carried from the center toward the periphery, we have

$$Q_m = 2\pi r \int_0^\infty \rho v_r dz = 2\pi r^3 \varepsilon \rho_\infty \omega \int_0^\infty F d\zeta. \quad (3.1)$$

But on the strength of the continuity equation we have

$$\int_0^\infty F d\zeta = -(1/3) W|_0^\infty = c/3$$

and from (3.1) we obtain

$$Q_m = 1.0497 \pi r^3 \omega \rho_\infty \varepsilon. \quad (3.2)$$

The frictional stress at the surface of the disk is expressed by the dependence

$$\tau_w = -\left(\mu \frac{\partial v_\theta}{\partial z}\right)_w = -AB_T r^2 \omega \varepsilon^{-1} G'(0).$$

The coefficient of the moment of the frictional forces is represented in the form

$$C_m = \frac{2\pi \int_0^r r^2 \tau_w dr}{(1/2)\rho_\infty \omega^2 r^5} = -\frac{4\pi}{5} \varepsilon G'(0) = 2.6777\varepsilon. \quad (3.3)$$

The heat flux from the surface of the disk to the gas which is analogous to this is equal to

$$q_w = -\left(\lambda \frac{\partial T}{\partial z}\right)_w = -\frac{\varkappa}{\sigma} AB_T^2 r^3 \varepsilon^{-1} Q'(0).$$

The Nusselt number (the integral heat-transfer characteristic) is represented by the expression

$$\mathrm{Nu} = \frac{r \int_0^r 2\pi r q_w dr}{\pi r^2 [(\lambda T)_w - (\lambda T)_\infty]} = -\frac{2}{5\varepsilon} \frac{Q'(0)}{Q_w^2 - 1} = 0.01238 \frac{1}{\varepsilon}. \quad (3.4)$$

A comparison of Eqs. (3.1)-(3.4) with the analogous expressions for the case of the rotation of a disk in an incompressible fluid [2] indicates that the allowance for compressibility does not alter the structure of the representation of the integral characteristics of the motion and heat transfer.

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